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UNIQUE EMBEDDINGS OF SIMPLE PROJECTIVE PLANE POLYHEDRAL MAPS

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ABSTRACT

We characterize the 3-valent polyhedral maps in the projective plane whose graphs have a unique embedding in the projective plane. This is done by demonstrating two forbidden subgraphs of the dual of these uniquely embeddable graphs.

1. Introduction

We say that a graph G has a unique embedding in a manifold M provided for any two embeddings of G in M a circuit C determines a face of one embedding if and only if it determines a face in the other embedding. It is a well-known theorem (see, for example, [2]) that the graphs of convex 3-dimensional polytopes (i.e. the planar 3-connected graphs) have unique embeddings in the plane (or sphere).

We say that a graph embedded in the projective plane is a *projective* plane *polyhedral map* or PPPM, if and only if each vertex has valence at least 3, the closure of each face in a 2-cell and no two faces have a multiply connected union. The PPPM's are thus an analog in the projective plane of the graphs of convex 3-dimensional polytopes.

The analogous theorem about unique embeddings does not hold. The complete graph on six vertices is a PPPM but obviously it is not uniquely embeddable. There are, however, many uniquely embeddable PPPM's. We shall characterize the uniquely embeddable simple (i.e. 3-valent) PPPM's.

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2. Definitions and preliminaries

We shall say that two embeddings of a PPPM are *isomorphic* provided a circuit in one is a face if and only if it is a face in the other. If all embeddings of G are isomorphic we say it has a *unique* embedding.

We shall say that two PPM's G and H are *isomorphic* provided there is a one-to-one function from the vertices edges and faces of G to the vertices, edges and faces of H preserving dimension and incidences.

Since a PPPM is a 2-cell embedding it will satisfy Euler's equation: $V - E + F = 1$. A well-known consequence of Euler's equation is

$$\sum (6 - i) p_i = 6 \tag{1}$$

which holds for all simple PPPM's, G, where p_i is the number of i-sided faces of G.

Suppose G is a PPPM and H is obtained from G by adding an edge e across a face of G where the vertices of e do not lie on the same edge of G. We say that H is obtained from G by *face splitting*. If e_1, e_2 and e_3 are consecutive edges of a face F_1 of G and if we add e across F with the endpoints on e_1 and e_3 we say that we have split *parallel* to e_2. Note that if we split parallel to e_2 we can obtain an isomorphic PPPM by splitting parallel to e_2 across the face F_2 which meets F_1 on e_2.

A splitting that creates a triangular face with 3-valent vertices will be called a truncation of v, where v is the vertex lying on the two edges to which the new vertices were added by the splittings.

Splittings done across a face F and subsequent splittings to the resulting faces obtained from F are said to be splittings *in the region* F. Also any splitting parallel to an edge of F will be considered to be a splitting in the region F.

The dual of face splitting is called vertex splitting. In the duals of the simple PPPM's it consists of choosing two edge disjoint paths which intersect on two vertices, lying on the boundary of the star of a vertex v, then replacing v by vertices v_1 and v_2, and finally joining v_1 to each vertex of one path and the vertices of v_2 to each vertex of the other path.

We say that a family F of PPPM's can be *generated* by face splitting (vertex splitting) from a set S of PPPM's if and only if for each $G \in F$ there is a $G_1 \in S$ and a sequence $G_1, \dots, G_n = G$ such that G_i is obtained from G_{i-1} by face splitting (vertex splitting) and each $G_i \in F$.

LEMMA 1. *If G is a uniquely embeddable PPPM and H is obtained from G by face splitting, then H is uniquely embeddable.*

PROOF. Let H have two non-isomorphic embeddings. Let us remove an edge $e = xy$ to produce two embeddings of G. These two embeddings are isomorphic. There is only one face of G across which we can add e to produce H because if there were two such faces F_1 and F_2 then F_1 and F_2 would meet at the points x and y. Since no two faces have a multiply connected union, F_1 and F_2 must meet on an edge e' in G and points x and y must belong to e' in G. But the edge xy is not an edge of G, thus e can be added across only one face of G. Now, however, adding e to the two isomorphic embeddings of G produces isomorphic embeddings of H. Thus the two embeddings of H are isomorphic. □

LEMMA 2. *The PPPM M_1 has exactly two embeddings as shown in Fig.* 1.

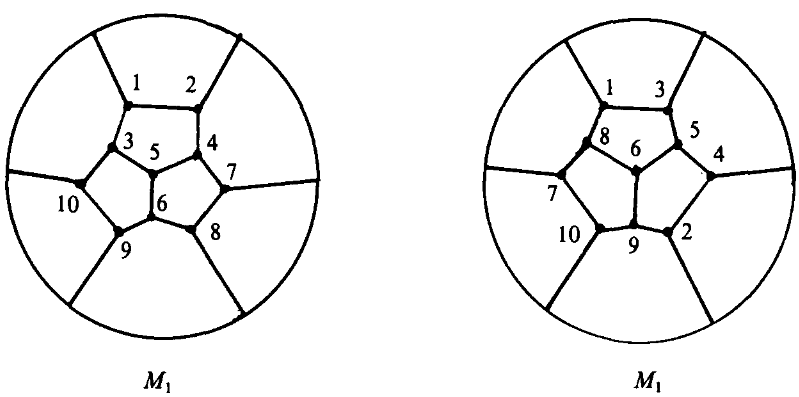
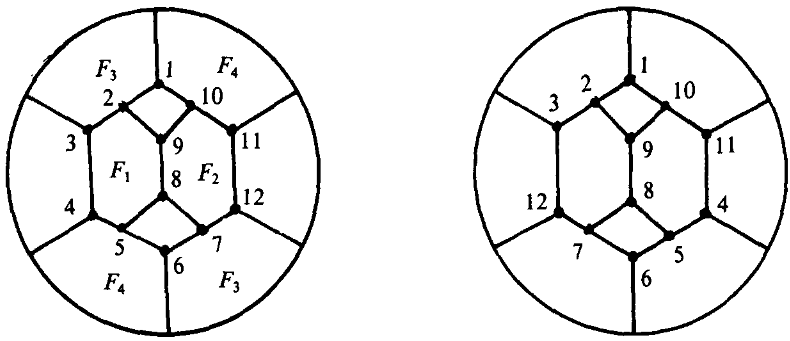


Fig. 1.

PROOF. By inspection one can determine that the circuits of length 5 that are not faces in the first embedding are 1–3–5–6–8, 2–4–5–6–9, 7–4–5–3–10, 7–8–6–9–10, 1–2–4–7–8 and 2–1–3–10–9. There are, however, the faces of the second embedding. Also as the reader may check, once one of these 5-circuits is required to be a face, the other faces are uniquely determined. This is easily checked because all embeddings of M_1 must have all 5-sided faces and thus all embeddings have the same facial structure. Once one face of a new embedding has its vertices embedded, then one can easily find the labels for the others uniquely. □

LEMMA 3. *The PPPM M_2 has exactly two embeddings as shown in Fig.* 2.



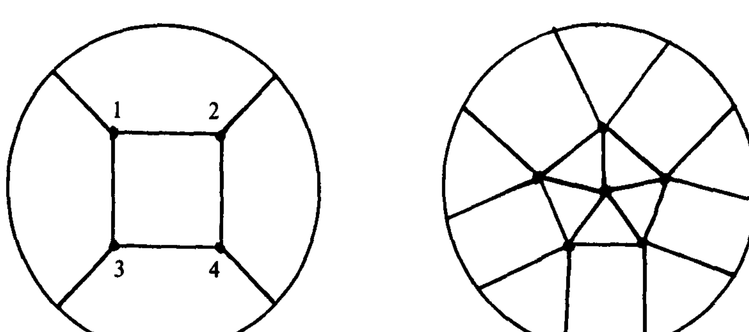
M_2 M_2

Fig. 2.

PROOF. The graph of M_2 is bipartite, thus in any embedding all faces are 4, 6, 8, ... sided. By Euler's equation we see that each embedding has seven faces. This rules out faces of 8 or more sides because an n-sided face will meet n distinct faces on its edges.

There are only three 4-circuits in the graph, thus by (1) they must be faces in any embedding. Since 1–10–9–2 and 7–8–5–6 are faces and 98 is an edge, in any embedding there is a 6-sided face containing either 2–9–8–7 or 2–9–8–5. Either choice uniquely determines the 6-sided face and then the 6-sided face containing 10–9–8 is determined. It is easily seen that once these four faces are determined the embedding is completely determined. □

THEOREM 1. *A simple PPPM is uniquely embeddable if and only if its dual does not contain the graphs G_1 or G_2 embedded as in Fig.* 3.



G_2 G_1

Fig. 3.

PROOF. The duals of the simple PPPM's are the triangulations of the projective plane. By a theorem of the author [1], these triangulations can be generated from the duals of M_1 and M_2 by vertex splitting, thus the simple PPPM's can be generated from M_1 and M_2 by face splitting.

Suppose G is a simple PPPM generated from M_1 by a sequence of face splittings. As we perform the splittings, the resulting PPPM's will have G_1 as a subgraph of their duals if the splittings either truncate vertices of M_1 or split faces inside the triangles resulting from such truncations (note that G is the dual of M_1).

Suppose after a sequence of splittings applied to M_1 we arrive at a PPPM M_3 whose dual M_3^* contains a copy of G_1. If we shrink edges of M_3^* we will never shrink an edge of that copy of G_1 because no edges of G_1 are shrinkable, thus M_3^* was generated by vertex splitting from that copy of G_1 contained in it. Thus the original copy of G_1 in the dual of M_1 remains unchanged during the face splittings. That is to say, no splittings other than truncations and splittings in the resulting triangular regions were performed. Thus a PPPM generated from M_1 by face splittings has G_1 in its dual if and only if all splittings are truncations and splittings in the resulting triangular regions. If we take two different embeddings of M_1 then clearly such splittings can be performed on both embeddings and thus G will not have a unique embedding.

Suppose, now, that some other kind of splitting is performed. In this case there will be a path P in G across a face F of M_1 concecting two nonadjacent edges of F. By the symmetry of M_1 we may assume the path is as in Fig. 4.

If G has two different embeddings, then by the argument in Lemma 1, they would have been generated from different embeddings of M_1. This, however, is

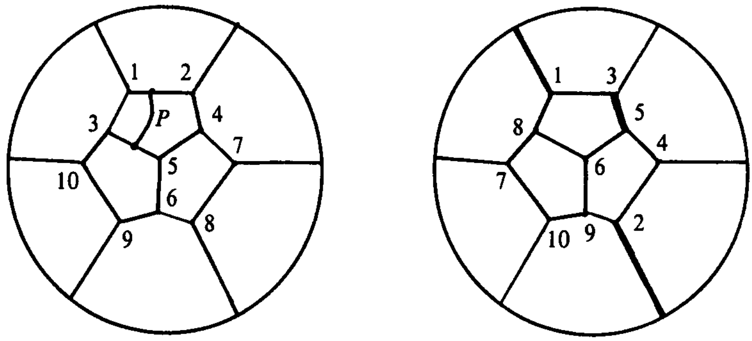


Fig. 4.

not possible because in the other embedding of M_1 the edges 12 and 35 do not lie on a common face (see Fig. 4). Thus in this case G has a unique embedding.

Next, suppose that G is generated from M_2. Note that the vertices of the dual of M_2 that correspond to the faces F_1, F_2, F_3 and F_4 of M_2 determine a subgraph of M_2^* with an embedding isomorphic to the embedding of G_2.

In this case the dual of G will also contain G_2 if all splittings are of three types:

(1) truncations of vertices M_2,

(2) subsequent splittings within the triangles created by the truncations in (1),

(3) splittings within the 4-sided faces of M_2 (recall the special meaning of splitting within a region with respect to parallel edges).

Suppose, after a sequence of splittings to M_2, we produce a PPPM, M_4, whose dual M_4^* contains G_2. If we shrink edges of M_4^* until we reach a minimal triangulation, we will never shrink an edge of the copy of G_2 in M_2^* because these edges are not shrinkable. Since the resulting minimal triangulations will be either G_1 or the dual of M_2 [1] and since G_1 does not contain G_2 as a submap, we arrive at the dual of M_2. The vertices of the copy of G_2 in the dual of M_2 will correspond to the 6-sided faces of M_2. Thus when the splittings are performed on M_2 the intersections of these 6-sided faces, or of faces resulting from splitting them are preserved. Thus the splitting will be of types (1), (2), or (3).

Such splittings can be done in both embeddings of M_2, thus in this case G does not have a unique embedding.

The proof now proceeds as in the case of M_1. If there is any other type of splitting applied to M_2, then there will be a path in G across a 6-sided face F of M_2 that is not parallel to an edge of a 4-sided face of M_2 and which joins nonadjacent vertices of M_2. The reader may check that the paths that exist in one embedding of M_2 don't exist in the other embedding. Thus any two embeddings of G are generated from only one embedding of M_2 and, by Lemma 1, G has a unique embedding.

References

1. D. Barnette, *Generating the triangulations of the projective plane*, J. Comb. Theory **33** (1982), 222–230.

2. W. T. Tutte, *How to draw a graph*, Proc. London Math. Soc. **13** (1963), 743–768.